C0 Type Shape Functions

The most common C0 type shape functions for one-dimensional structures are the linear functions

$$S_1 = 1 - x \qquad S_2 = x$$

for the interval [0,1]. It is desired that the higher order functions vanish at the end points $x=0$ and $x=1$ so that the function to be expanded has the value at $x=0$ of the coefficient of S_1 and the value at $x=1$ of the coefficient of S_2. A set of orthogonal, hierarchical functions that satisfies these conditions is given by

$$S_{n+3} = x(1-x)G_n(5,3,x) \qquad (n=0,1,\ldots)$$

These functions vanish at the end points and are hierarchical in the sense described earlier. Here, $G_n(5,3,x)$ are Jacobi polynomials[3] which are orthogonal over the interval [0,1] with the weighting function $x^2(1-x)^2$. Thus, C0 type shape functions S_i are orthogonal to S_j when i and j are greater than 2 and not equal to each other. Since $G_n(5,3,x)$ is a polynomial of degree n, the C0 function S_{n+3} is of degree $n+2$ for $n=0,1,\ldots$. In Ref. 3 the recurrence relations that are needed to generate Jacobi polynomials are given.

C1 Type Shape Functions

The most common C1 type shape functions for one-dimensional structures are the cubic functions

$$S_1 = 1 - 3x^2 + 2x^3 \qquad S_2 = x - 2x^2 + x^3$$

$$S_3 = 3x^2 - 2x^3 \qquad S_4 = -x^2 + x^3$$

for the interval [0,1]. It is desired that the higher order functions and their first derivatives vanish at the end points $x=0$ and $x=1$. This is so that the function to be expanded and its first derivative have the values at $x=0$ of the coefficients of S_1 and S_2, respectively. Similarly, the function and its first derivative have the values at $x=1$ of the coefficients of S_3 and S_4, respectively. A set of orthogonal, hierarchical functions that satisfies these conditions is given by

$$S_{n+5} = x^2(1-x)^2 G_n(9,5,x) \qquad (n=0,1,\ldots)$$

These functions and their first derivatives vanish at the end points and are hierarchical in the sense described earlier. Here, $G_n(9,5,x)$ are Jacobi polynomials[3] which are orthogonal over the interval [0,1] with the weighting function $x^4(1-x)^4$. Thus, C1 type shape functions S_i are orthogonal to S_j when i and j are greater than 4 and not equal to each other. Since $G_n(9,5,x)$ is a polynomial of degree n, the C1 function S_{n+5} is of degree $n+4$ for $n=0,1,\ldots$.

Discussion

Linear dynamic results obtained from the present shape functions are identical to those obtained in Ref. 4 for the C0 case and identical to those obtained in Refs. 1,5, and 6 for the C1 case. In Refs. 4-6 the element displacement is expanded in a simple power series for which conditioning problems are known to occur. The element matrices obtained from the present shape functions, however, are so well conditioned, that polynomials of order 15 and higher have been used in the identical manner as in Ref. 1 with virtually no sign of ill conditioning on a 60-bit-word computer with single precision (14-place) arithmetic. In addition, the present shape functions lead to sparse, hierarchical matrices. The orthogonality of the higher order shape functions contributes to the sparsity and overall good conditioning of these matrices.

The higher order shape functions presented herein are a special case of general higher order shape functions in that they (and their first derivatives in the case of C1 type functions) vanish at the element boundaries. This implies that their corresponding generalized coordinates can be eliminated from the global equations in static problems. This fact can be used to significant advantage, particularly in nonlinear problems, in that equations for the generalized coordinates associated with each element can be solved at the element level, generally leading to a savings in storage and computer operations.

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Extrapolation of Optimum Design Based on Sensitivity Derivatives

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Introduction

IN many applications, it is useful to know how the perturbations of the design problem constants (parameters) will affect the problem optimum solution. For this purpose, optimum sensitivity analysis has been proposed in Ref. 1 as a structural design tool to assess the parameter perturbation effects by extrapolations based on optimum sensitivity derivatives, without resorting to relatively costly reoptimization. With respect to the extrapolation accuracy, the numerical experience in structural applications reported to date[1,2] showed that, in most cases, accuracy sufficient for engineering purposes is attainable over a range in excess of 20% of the parameter value.

The purpose of this Note is to contribute to the aforementioned experience a test case deliberately chosen so as to have a strong nonlinearity of the constraints, with

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respect to the design variables, in order to test the optimum-based sensitivity under conditions more adverse than those reported to date. The example corresponds to a typical aerospace element: the composite sandwich panel under in-plane loads. The underlying theoretical developments are given in Ref. 1.

The Optimization Problem

The symmetric sandwich panel shown in Fig. 1 is made of a relatively thick aluminum honeycomb core and two graphite-epoxy face sheets. Each face sheet consists of layers oriented at 0 deg, $\pm\phi$, and 90 deg, uniformly dispersed through the thickness. The mass of panel is to be minimized with respect to the thickness of the different layers and the total panel thickness (t_0, $t_\phi = t_{+\phi} = t_{-\phi}$, t_{90}, and h). The design is subjected to constraints which limit the stresses governing a fiber failure mode, limit the stresses governing a matrix failure mode, limit the strains in the panel, guard against buckling, require positive thickness of the core material, and establish lower or upper bounds for the variables. Stresses and strains are obtained by the classical lamination theory and the optimization is performed by the method of usable-feasible directions. A detailed description of the problem is given in Ref. 3.

The specific data used in this problem are presented in Table 1 and the values of the optimum design variables and mass are given in Table 2. At the optimum point, the constraints on strain in the Y direction, on shear strain, on buckling, and the lower bound on variable t_0 are active.

Optimum Sensitivity Analysis

The results presented pertain to sensitivity analysis of the optimum solution described above with respect to the angle ϕ. Sensitivity derivatives obtained using the approach based on the Lagrange multiplier equations are given in Table 2.

Table 1 Data for the initial optimum problem

Geometrical data
$a = 50$ cm, $b = 36$ cm, $\phi = 45$ deg
Face sheet material
$E_{11} = 14{,}700{,}000$ N/cm^2, $E_{22} = 10{,}900{,}000$ N/cm^2,
$G_{12} = 641{,}000$ N/cm^2
$\nu_{12} = 0.38$, $\rho = 1.61$ g/cm^3
$X_t = 146{,}000$ N/cm^2, $X_c = 141{,}000$ N/cm^2
$Y_t = 4200$ N/cm^2, $Y_c = 14{,}800$ N/cm^2
$S = 9520$ N/cm^2
Allowable strains = 0.004 cm/cm
Nonstructural core material
$\rho = 0.09$ g/cm^3
Applied Loads
$N_x = -5000$ N/cm, $N_y = -10{,}000$ N/cm, $N_{xy} = 2000$ N/cm
Geometrical constraints
$0.015 \le t \le 1.0$ cm, $t = t_0, t_\phi, t_{90}$
$0.06 \le h \le 4.0$ cm
$t_0 + 2t_\phi + t_{90} \le 0.5$ h

Table 2 Optimum design and sensitivity derivatives with respect to the angle ϕ ($\phi = 45$ deg)

Variables	Optimal value, cm	Sensitivity derivative, cm/rad
t_0	0.0150	0.0
t_ϕ	0.0561	$1.3864\ 10^{-2}$
t_{90}	0.1033	$-1.6354\ 10^{-1}$
h	1.4305	$1.3232\ 10^{-1}$
Objective M	862.3 g	-3.5015×10^2 g/rad

Combining the values of the optimum design variables and objective function with their sensitivity derivatives (Table 2), one may use linear extrapolation to approximate those optimum characteristics over a limited range of variation of the parameter ϕ. In Fig. 1 the linear approximations are compared to the actual curves obtained by optimizations for different values of the parameter. The tangency between the two sets of curves at the initial point reflects the accuracy of the sensitivity derivatives. For a 20% decrement in the angle ϕ, the variable t_ϕ and the objective function M are underestimated by only 9.1 and 1.5%, respectively, while those differences are 21.4 and 4.1% for a comparable increment. Also, as ϕ is decreased by 20%, the predicted changes in t_ϕ and M are respectively -69 and 79% of the actual changes, while as ϕ is increased by the same amount those predictions are 12 and 263%. The poor prediction of the changes in t_ϕ when the angle is decreased should not be a matter of too much concern since that variable varies little in that direction.

Clearly, in this specific example, the quality of the extrapolations is significantly better when the parameter is decreased than when it is increased. This may be traced to the fact that as the angle is increased, the set of constraints binding the optimum design changes. Indeed, an optimization for 10% increment in the angle ϕ reveals that the constraint on the strain in shear which was active at the original optimum ceases to be active and is replaced by the constraint on the strain in the X direction. This must be accompanied by a degradation of the quality of the extrapolations since the equations giving the sensitivity derivatives are derived on the assumption that the set of active constraints does not change as the parameter varies. As the angle is decreased by up to 20%, no change occurs in the active constraints set; therefore, better approximations are obtained. A change in active constraints set was mentioned in Ref. 1 as a possible source of inaccuracy for extrapolations based on sensitivity derivatives; however, no example of such a situation has been described prior to this Note.

In Ref. 1 equations are proposed to evaluate by linear extrapolation the parameter increments which render a previously inactive constraint active or a previously active constraint redundant. Their use in this particular example reveals that change-of-status predictions should be interpreted with caution. Indeed, those equations predict that the inactive constraint on strain in the X direction should become active for a small (6%) increment in the angle, as confirmed by the

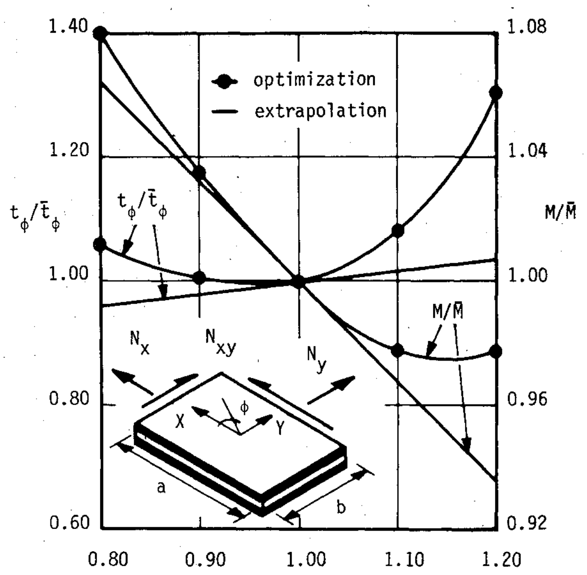

Fig. 1 Optimum mass M and thickness t_ϕ as functions of the angle ϕ (overbars denote quantities at initial optimum (Table 2); parameters defined in inset are given in Table 1).

optimization results reported above. In contrast, they also predict that the active constraint on shear strain should become redundant for a large (85%) increment in the angle, while, as noted previously, it becomes inactive for an increment of only 10%.

In the course of this study, sensitivity analyses of the optimum solution were performed with respect to a number of different parameters, including applied loads and geometrical or material characteristics. Among all the parameters tested, the angle ϕ was chosen to be reported here in detail for its most nonlinear influence on the optimum solution.

Conclusions

In this Note, an example of optimum sensitivity analysis of composite panel is discussed. The case was deliberately chosen for the nonlinearity of the constraints and, also, the tendency for the set of active constraints to change when the parameters are perturbed. Under those adverse conditions, sensitivity derivatives are used in linear extrapolations to extend optimization information over a limited range of variation of a typical parameter.

Comparison with reoptimization results verifies the accuracy of the optimum derivatives predictions but shows that large errors may be observed when a parameter variation causes changes in the set of active constraints. It is therefore necessary to be able to foresee those changes if optimum sensitivity analyses are to be performed routinely during the execution of complex optimization procedures. As shown here, the existing methods for estimating the parameter variation causing such changes can be unreliable.

In the absence of changes in the active constraints set, linear approximations appear to yield reasonable accuracy for significant variations in the parameter, as seen in the left half of Fig. 1; the prediction is often quite a bit better for the objective function than for the design variables. Comparable observations were made in Ref. 1.

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